## FIXED INCOME

## APPENDIX - DERIVATION OF THE FULL TERM STRUCTURE MFM AND THE SWIFT, TWIST AND BUTTERFLY MFM

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2. Appendix: Derivation of the full term structure MFM and the shift, twist and butterfly MFM

In this appendix we present the mathematical derivation of both multifactor models.

### 1.1 Full term structure MFM

We can start from the conventional evaluation of a bond model:

$$
\begin{align*}
& \mathrm{P}_{\mathrm{i}}^{\mathrm{t}}=\sum_{\mathrm{j}=1}^{\mathrm{T}} \mathrm{CF}_{\mathrm{i}}^{\mathrm{t+j}} \cdot \mathrm{PDB}_{\mathrm{t}+\mathrm{j}}^{\mathrm{t}}  \tag{1}\\
& \\
& \mathrm{PDB}_{\mathrm{t}+\mathrm{j}}^{\mathrm{t}}=\frac{1}{\left(1+\mathrm{k}_{\mathrm{t}+\mathrm{j}}^{\mathrm{t}}\right)^{\mathrm{j}}}
\end{align*}
$$

where:

| $\mathrm{P}_{\mathrm{i}}^{\text {t }}$ | is the price of a coupon bond $i$ at time $t$, |
| :---: | :---: |
| $\mathrm{CF}_{\mathrm{i}}^{\text {t+j }}$ | is the cash-flow of bond i at time $\mathrm{t}+\mathrm{j}$, |
| $\mathrm{PDB}_{\mathrm{t}+\mathrm{j}}^{\mathrm{t}}$ | is the price at time $t$ of a default-risk-free discount bond maturing at time $t+j$, |
| $\mathrm{k}_{\mathrm{t}+\mathrm{j}}^{\mathrm{t}}$ | is the yield at time t of a default-risk-free discount bond maturing at time $\mathrm{t}+\mathrm{j}$. |

Using (1) for instantaneous movements in the discount rate term structure and extending it as a proxy for price variations through small time periods, we get:

$$
\Delta \mathrm{P}_{\mathrm{i}}^{\mathrm{t},+1}=\sum_{\mathrm{j}=1}^{\mathrm{T}} \mathrm{CF}_{\mathrm{i}}^{\mathrm{t}+\mathrm{j}} \cdot \Delta \mathrm{PDB}_{\mathrm{t}+\mathrm{j}}^{\mathrm{t}+1}
$$

Dividing this last equation by the price of the bond and multiplying and dividing by $\mathrm{PDB}_{\mathrm{t}+\mathrm{j}}^{\mathrm{t}}$ we have:

$$
\frac{\Delta \mathrm{P}_{\mathrm{i}}^{\mathrm{t}+\mathrm{t}+1}}{\mathrm{P}_{\mathrm{i}}^{\mathrm{t}}}=\sum_{\mathrm{j}=1}^{\mathrm{T}} \frac{\mathrm{CF}_{\mathrm{i}}^{\mathrm{t+j}} \cdot \mathrm{PDB}_{\mathrm{t}+\mathrm{j}}^{\mathrm{t}}}{\mathrm{P}_{\mathrm{i}}^{\mathrm{t}}} \cdot \frac{\Delta \mathrm{PDB}_{t+j}^{\mathrm{t}+1}}{\mathrm{PDB}_{\mathrm{t}+\mathrm{j}}^{\mathrm{t}}}
$$

Introducing some new notations, the previous equation can be shown as the full term structure MFM equation:

$$
\begin{equation*}
R_{i}^{t, t+1}=\sum_{j=1}^{T} z_{i}^{t+j} \cdot \text { RDB }_{t+j}^{t, t+1} \quad \text { with } \quad \sum_{j=1}^{T} z_{i}^{t+j}=1 \tag{2}
\end{equation*}
$$

where:
$R_{i}^{t, t+1}=\frac{\Delta P_{i}^{t, t+1}}{P_{i}^{t}} \quad$ is the return of the bond $i$ between $t$ and $t+1$,
$z_{i}^{t+j}=\frac{\mathrm{CF}_{i}^{t+j} \cdot \mathrm{PDB}_{t+j}^{t}}{P_{i}^{t}}$ is the fraction of the value of the bond $i$ related to cash flow $t+j$,
$\mathrm{RDB}_{t+j}^{\mathrm{tt+1}}=\frac{\Delta \mathrm{PDB}_{t+j}^{t, t+1}}{\operatorname{PDB}_{t+j}^{t}}$ is the return of the default-risk free discount bond maturing at $\mathrm{t}+\mathrm{j}$ between t and $\mathrm{t}+1$.

From equation (2), we can analyse the risk of coupon bond i by calculating the variance of its returns using the following equation: ${ }^{1}$

$$
\mathrm{V}\left(\mathrm{R}_{\mathrm{i}}^{\mathrm{t}+t+1}\right)=\sum_{j=1}^{T} \sum_{n=1}^{T} z_{i}^{t+j} \cdot z_{i}^{t+n} \cdot \operatorname{Cov}\left(R D B_{t+j}^{t, t+1}, R D B_{t+n}^{t, t+1}\right)
$$

where:
$\mathrm{V}\left(\mathrm{R}_{\mathrm{i}}^{\mathrm{t}+1+1}\right) \quad$ is the variance of the return of bond i between time t , and time $\mathrm{t}+1$, $\operatorname{Cov}\left(\mathrm{RDB}_{t+j}^{\mathrm{t}++1}, \mathrm{RDB}_{t+n}^{\mathrm{t}+1+1}\right)$ is the covariance of the returns of the default-risk free discount bonds maturing at $\mathrm{t}+\mathrm{j}$ and $\mathrm{t}+\mathrm{n}$ between t and $\mathrm{t}+1$.

As the changes of the yields of the different discount bonds are not independent, the variance of the returns of a coupon bond $i$ is the sum of the covariances of each pair of discount bonds weighted by the factor exposures of bond i . When both discount bonds have the same maturity, that is $j=n$, instead of a covariance, we have the variance of the returns of the discount bond with this maturity. Both historical estimates and forecasts of these covariances can be obtained from time series analysis of the factor returns.

[^0]
### 1.2 The shift, twist and butterfly MFM

We assume that:

$$
\begin{equation*}
\mathrm{RDB}_{\mathrm{t}+\mathrm{j}}^{\mathrm{t}+1}=\mathrm{f}_{\mathrm{j}} \cdot \mathrm{RS}^{\mathrm{t}, \mathrm{t}+1}+\mathrm{g}_{\mathrm{j}} \cdot \mathrm{RT}^{\mathrm{t}, \mathrm{t}+1}+\mathrm{h}_{\mathrm{j}} \cdot \mathrm{RB}^{\mathrm{t}, \mathrm{t}+1} \tag{3}
\end{equation*}
$$

where:
RS $^{\mathrm{t}, t+1} \quad$ is the return of the shift factor between $t$ and $t+1$, $\mathrm{RT}^{\mathrm{t}, \mathrm{t}+1} \quad$ is the return of the twist factor between t and $\mathrm{t}+1$,
$R^{t, t+1} \quad$ is the return of the butterfly factor between $t$ and $t+1$.
And where the coefficients $f_{j}, g_{j}$ and $h_{j}$ are defined by ${ }^{2}$ :

|  | Short <br> Maturities (j) | Intermediate <br> Maturities $(\mathrm{j})$ | Long <br> Maturities $(\mathrm{j})$ |
| :---: | :---: | :---: | :---: |
| $\mathrm{f}_{\mathrm{j}}$ | +a | +a | +a |
| $\mathrm{g}_{\mathrm{j}}$ | +b | 0 | -b |
| $\mathrm{h}_{\mathrm{j}}$ | +c | -2 c | +c |

where $\mathrm{a}, \mathrm{b}$ and c are all parameters that can be estimated econometrically.
Using equations (2) and (3), we can arrive to the equation of this approach:

$$
\mathrm{R}_{\mathrm{i}}^{\mathrm{t}, \mathrm{t}+1}=\mathrm{zs}_{\mathrm{i}}^{\mathrm{t}} \cdot \mathrm{RS}^{\mathrm{t},+1}+\mathrm{zt}_{\mathrm{i}}^{\mathrm{t}} \cdot \mathrm{RT}^{\mathrm{t}, \mathrm{t}+1}+\mathrm{zb}_{\mathrm{i}}^{\mathrm{t}} \cdot \mathrm{RB}^{\mathrm{t}, \mathrm{t}+1}
$$

where:
$z s_{i}^{t}=\sum_{j=1}^{T} z_{i}^{t+j} \cdot f_{j}$ is the exposure of bond $i$ to the shift factor,
$z t_{i}^{t}=\sum_{j=1}^{T} z_{i}^{t+j} \cdot g_{j}$ is the exposure of bond $i$ to the twist factor,
$z b_{i}^{t}=\sum_{j=1}^{T} z_{i}^{t+j} \cdot h_{j}$ is the exposure of bond $i$ to the butterfly factor.
As in the previous section, we can express the variance of the returns of bond ias: ${ }^{3}$

$$
\begin{aligned}
\mathrm{V}\left(\mathrm{R}_{\mathrm{i}}^{\mathrm{t}++1}\right)= & \left(z s_{i}^{t}\right)^{2} \cdot V\left(R S^{t, t+1}\right)+\left(z t_{i}^{t}\right)^{2} \cdot V\left(R T^{t, t+1}\right)+\left(z b_{i}^{t}\right)^{2} \cdot V\left(R B^{t, t+1}\right) \\
& +2 \cdot z s_{i}^{t} \cdot \mathrm{zt}_{i}^{t} \cdot \operatorname{Cov}\left(R S^{t, t+1}, R T^{t, t+1}\right)+2 \cdot z s_{i}^{t} \cdot \mathrm{zb}_{i}^{t} \cdot \operatorname{Cov}\left(R S^{t, t+1}, R B^{t, t+1}\right)+2 \cdot z t_{i}^{t} \cdot \mathrm{zb}_{i}^{t} \cdot \operatorname{Cov}\left(R T^{t, t+1}, R B^{t, t+1}\right)
\end{aligned}
$$

where:
$\mathrm{V}\left(\mathrm{RS}^{\mathrm{t}, t+1}\right) \quad$ is the variance of the return of the shift factor between t and $\mathrm{t}+1$,

[^1]$\mathrm{V}\left(\mathrm{RT}^{\mathrm{t},+1}\right) \quad$ is the variance of the return of the twist factor between t and $\mathrm{t}+1$,
$\mathrm{V}\left(\mathrm{RB}^{\left.\mathrm{t}{ }^{t+1}\right)}\right.$ is the variance of the return of the butterfly factor between $t$ and $t+1$,
$\operatorname{Cov}\left(\mathrm{RS}^{\mathrm{t},+1}, \mathrm{RT}^{\mathrm{t},+1}\right)$ is the covariance of the returns of the shift and twist factors,
$\operatorname{Cov}\left(\mathrm{RS}^{\mathrm{t},+1}, \mathrm{RB}^{\mathrm{t},+1}\right)$ is the covariance of the returns of the shift and butterfly factors,
$\operatorname{Cov}\left(\mathrm{RT}^{\mathrm{t},+1}, \mathrm{RB}^{\mathrm{t}+1+1}\right)$ is the covariance of the returns of the twist and butterfly factors.
As in the previous approach, the variance of the coupon bond $i$ is the sum of the variances and covariances of each pair of factors. Here, we only have three factors, so only three variances and three covariances have to be forecasted to calculate the variance of the coupon bond i in order to have a predictive bond risk model. The weighted covariances are multiplied by two because by definition $\operatorname{Cov}(x, y)=\operatorname{Cov}(y, x)$ for any $x$ and $y$.


[^0]:    1 If we use a vector notation, we have $\mathrm{z}_{\mathrm{i}}^{\mathrm{t}}=\left(\mathrm{z}_{\mathrm{i}}^{\mathrm{t}+3 \mathrm{~m}}, \mathrm{z}_{\mathrm{i}}^{\mathrm{t}+6 \mathrm{~m}}, \mathrm{z}_{\mathrm{i}}^{\mathrm{t}+1 \mathrm{y}}, \ldots, \mathrm{z}_{\mathrm{i}}^{\mathrm{t}+15 \mathrm{y}}\right)$ and $\mathrm{V}\left(\mathrm{R}_{\mathrm{i}}^{\mathrm{t}+1+1}\right)=\mathrm{z}_{\mathrm{i}}^{\mathrm{t}^{\prime}} \cdot \mathrm{W} \cdot \mathrm{z}_{\mathrm{i}}^{\mathrm{t}}$.

[^1]:    ${ }^{2}$ For the reason of simplicity, we define here the coefficients $f_{j}, g_{j}$, and $h_{j}$ arbitrarily. This can be done more efficiently using the principal component statistical method to determine the first three components. It is interesting to note that using this statistical procedure provides similar interpretation about the basic movements of the term structure, which were implied by the shift, twist and butterfly factors.
    3 Using a vector notation: $\mathrm{V}\left(\mathrm{R}_{\mathrm{i}}^{\mathrm{t}+1}\right)=\left(\mathrm{zs}_{\mathrm{i}}^{\mathrm{t}}, \mathrm{zt}_{\mathrm{i}}^{\mathrm{t}}, \mathrm{zb}_{\mathrm{i}}^{\mathrm{t}}\right)^{\prime} \cdot \mathrm{W} \cdot\left(\mathrm{zs}_{\mathrm{i}}^{\mathrm{t}}, \mathrm{zt}_{\mathrm{i}}^{\mathrm{t}}, \mathrm{zb}_{\mathrm{i}}^{\mathrm{t}}\right)$.

